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# Multi-instanton calculus in N=2 supersymmetric QCD

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## Abstract

Microscopic tests of the exact results are performed in N=2 SU(2) supersymmetric QCD. We construct the multi-instanton solution in N=2 supersymmetric QCD and calculate the two-instanton contribution  $\mathcal{F}_2$  to the prepotential  $\mathcal{F}$  explicitly. For  $N_f = 1, 2$ , instanton calculus agrees with the prediction of the exact results, however, for  $N_f = 3$ , we find a discrepancy between them.

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# 1 Introduction

Recently, much progress has been made in the study of the strongly coupled supersymmetric gauge theories. Under the holomorphy and the duality, the low energy effective actions of N=2 supersymmetric Yang-Mills theory and supersymmetric QCD in Coulomb phase are determined exactly for SU(2) gauge group[1] and later for larger gauge groups[2]-[6]. These low energy effective theories reveal the interesting results like the monopole condensation[1] and new supersymmetric conformal field theories[7, 8] and so on.

The exact results predict the non-perturbative corrections from instanton. Furthermore, it is known that the instanton calculus in the supersymmetric theories is fully controllable when the theories is weakly coupled [9, 10, 11]. Therefore, the instanton calculus gives a non-trivial test of the exact results. Until now, the instanton calculi were performed in the pure Yang-Mills theories and all the microscopic calculi agree with the exact results [12, 13, 14, 15].

In this letter, we examine the consistencies between the instanton calculus and the exact results of N=2 supersymmetric QCD. Especially, we focus on the N=2 supersymmetric SU(2) QCD. In N=2 supersymmetric SU(2) QCD, there is a parity symmetry between hypermultiplets, then only contributions from even number of instanton exist [1]. Thus the instanton corrections start from the two-instanton sector. In the following, we perform the two-instanton calculus in N=2 supersymmetric SU(2) QCD for  $N_f \leq 3$  flavors<sup>1</sup> and compare it with the exact results.

## 2 The construction of multi-instanton

First we will briefly summarize N=2 vector multiplet of supersymmetric instanton[14]. The defining equations of N=2 vector multiplet of supersymmetric instanton are following;

$$F_{\mu\nu} = -\tilde{F}_{\mu\nu}, \tag{1}$$

$$\bar{D}\lambda = 0, \quad \bar{D}\psi = 0, \tag{2}$$

$$D^2\phi - \sqrt{2}i[\lambda, \psi] = 0. \tag{3}$$

When the coupling constant is small enough, the solution of the above equation dominates the path integral. In supersymmetric theories, the coupling dependence of the instanton

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<sup>1</sup>In the following, we give the supersymmetric instanton in the two-instanton sector. However, the construction is applicable to the arbitrary number of multi-instantons. More details are given in [24].

contribution is fixed, then it is enough to consider the small coupling case. The first three equations in the above mean that  $A_\mu$  is an instanton and  $\lambda$  and  $\psi$  are adjoint fermion zero modes. The last one is the supersymmetric version of the 't Hooft equation[16]. The multi-instanton solution is constructed by ADHM method[17, 18]. In the two-instanton sector, its explicit form is the following;

$$A_\mu = iN^{\dagger\dot{r}r}\partial_\mu N_{r\dot{s}}, \quad (4)$$

where  $N$  is a quaternionic 3-dimensional column vector <sup>2</sup> obeying

$$N^\dagger M = 0, \quad N^\dagger N = 1. \quad (5)$$

Here  $M$  is a  $3 \times 2$  matrix made up of quaternions;

$$M = \begin{pmatrix} \omega_1 & \omega_2 \\ x_0 - x + a_3 & a_1 \\ a_1 & x_0 - x - a_3 \end{pmatrix}, \quad (6)$$

$$a_1 = \frac{a_3}{4|a_3|^2} (\bar{\omega}_2 \omega_1 - \bar{\omega}_1 \omega_2).$$

The relation between  $a_1$  and  $a_3$  is required by the reality condition of  $R = M^\dagger M$  and this ensures the anti-self-duality of  $F_{\mu\nu}$ . The adjoint fermionic zero modes are the following[19];

$$\lambda_{\alpha\dot{s}}^{\dot{r}} = N^{\dagger\dot{r}r} \left\{ \mathcal{M}_r R^{-1} C^T \delta_\alpha^s + \epsilon_{r\alpha} C R^{-1} (\mathcal{M}^T)^s \right\} N_{s\dot{s}}, \quad (7)$$

$$\psi_{\alpha\dot{s}}^{\dot{r}} = N^{\dagger\dot{r}r} \left\{ \mathcal{N}_r R^{-1} C^T \delta_\alpha^s + \epsilon_{r\alpha} C R^{-1} (\mathcal{N}^T)^s \right\} N_{s\dot{s}}, \quad (8)$$

where

$$\mathcal{M}_s = \begin{pmatrix} \mu_{1s} & \mu_{2s} \\ 4\xi_s + m_{3s} & m_{1s} \\ m_{1s} & 4\xi_s - m_{3s} \end{pmatrix}, \quad \mathcal{N}_s = \begin{pmatrix} \nu_{1s} & \nu_{2s} \\ 4\xi'_s + n_{3s} & n_{1s} \\ n_{1s} & 4\xi'_s - n_{3s} \end{pmatrix}, \quad (9)$$

$$m_1 = \frac{a_3}{2|a_3|^2} (2\bar{a}_1 m_3 + \bar{\omega}_2 \mu_1 - \bar{\omega}_1 \mu_2), \quad (10)$$

$$n_1 = \frac{a_3}{2|a_3|^2} (2\bar{a}_1 n_3 + \bar{\omega}_2 \nu_1 - \bar{\omega}_1 \nu_2), \quad (11)$$

$$C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (12)$$

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<sup>2</sup>We represent a quaternion  $N$  as  $2 \times 2$  matrix  $N_{s\dot{s}} = -iN_\mu \sigma_{\mu s\dot{s}}$

The solution of Eq.(3) is a sum of the solution of the homogeneous equation  $D^2\phi_0 = 0$  and a particular solution  $\phi_f$ ;

$$\phi = \phi_0 + \phi_f. \quad (13)$$

The explicit forms of  $\phi_0$  and  $\phi_f$  are

$$\phi_0 = -iN^{\dagger\dot{r}r}A_r^sN_{s\dot{s}}, \quad (14)$$

$$\phi_f = \frac{\sqrt{2}i}{4}N^{\dagger\dot{r}r}\left\{\mathcal{N}_rR^{-1}(\mathcal{M}^T)^s - \mathcal{M}_rR^{-1}(\mathcal{N}^T)^s + iF\delta_r^s\right\}N_{s\dot{s}}, \quad (15)$$

where

$$A_r^s = \begin{pmatrix} A_{00r}^s & 0 & 0 \\ 0 & 0 & \gamma\delta_r^s \\ 0 & -\gamma\delta_r^s & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & -\alpha & 0 \end{pmatrix}, \quad (16)$$

$$A_{00} = i\langle\phi\rangle, \quad \gamma = -\frac{\omega}{H}, \quad (17)$$

$$\alpha = -\frac{i}{H}(\mu_1\nu_2 - \mu_2\nu_1 + 2m_3n_1 - 2m_1n_3), \quad (18)$$

$$L = |\omega_1|^2 + |\omega_2|^2, \quad H = L + 4|a_1|^2 + 4|a_3|^2, \quad (19)$$

$$\Omega = \omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1, \quad \omega = \frac{1}{2}\text{tr}(\Omega A_{00}).$$

In N=2 supersymmetric QCD, there appear  $N_f$  hypermultiplets in the theory. The N=2 hypermultiplets of supersymmetric instanton are characterized by the following equations;

$$\bar{D}q = 0, \quad \bar{D}\tilde{q} = 0, \quad (20)$$

$$D^2Q - \sqrt{2}i\lambda q = 0, \quad D^2\tilde{Q} + \sqrt{2}i\tilde{q}\lambda = 0, \quad (21)$$

$$D^2Q^\dagger - \sqrt{2}i\tilde{q}\psi = 0, \quad D^2\tilde{Q}^\dagger - \sqrt{2}i\psi q = 0. \quad (22)$$

The first two equations indicate that  $q$  and  $\tilde{q}$  are the fundamental fermionic zero modes[20];

$$q_{f\alpha}^{\dot{r}} = \Psi_\alpha^{\dot{r}}\zeta_f, \quad \tilde{q}_{f\dot{r}}^\alpha = -\epsilon^{\alpha\beta}\epsilon_{\dot{r}\dot{s}}\Psi_{\beta l}^{\dot{s}}\tilde{\zeta}_f, \quad (23)$$

where the indices  $\alpha$ ,  $\dot{r}$  and  $f$  of  $q$  and  $\tilde{q}$  are a spinor, color and flavor index respectively.  $\Psi$  is a following normalized function.

$$\Psi_\alpha^{\dot{r}} = -\frac{1}{\pi}N^{\dagger\dot{r}r}\epsilon_{r\alpha}CR^{-1}, \quad \int d^4x\epsilon^{\alpha\beta}\epsilon_{\dot{r}\dot{s}}\Psi_{\alpha k}^{\dot{r}}\Psi_{\beta l}^{\dot{s}} = -\delta_{kl}. \quad (24)$$

The solution of Eq.(21) and (22) is given by

$$Q_f^{\dot{r}} = \frac{\sqrt{2}i}{4\pi}N^{\dagger\dot{r}r}\mathcal{M}_rR^{-1}\zeta_f, \quad \tilde{Q}_{f\dot{r}} = -\frac{\sqrt{2}i}{4\pi}\epsilon_{\dot{r}\dot{s}}N^{\dagger\dot{s}r}\mathcal{M}_rR^{-1}\tilde{\zeta}_f, \quad (25)$$

$$Q_{f\dot{r}}^\dagger = \frac{\sqrt{2}i}{4\pi}\epsilon_{\dot{r}\dot{s}}N^{\dagger\dot{s}r}\mathcal{N}_rR^{-1}\tilde{\zeta}_f, \quad \tilde{Q}_f^{\dagger\dot{r}} = \frac{\sqrt{2}i}{4\pi}N^{\dagger\dot{r}r}\mathcal{N}_rR^{-1}\zeta_f. \quad (26)$$

In N=2 supersymmetric QCD, the anti-scalar component of N=2 vector multiplet satisfies the following equation.

$$D^2\phi^{\dagger a} - \sqrt{2}i\tilde{q}T^a q = 0. \quad (27)$$

The solution of Eq.(27) is given by,

$$\phi^\dagger = \phi_0^\dagger + \phi_q^\dagger, \quad (28)$$

$$\phi_q^\dagger = -iN^{\dagger rr}PN_{r\dot{s}}, \quad (29)$$

where

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & -\beta & 0 \end{pmatrix}, \quad \beta = \frac{\sqrt{2}}{16} \frac{\tilde{\zeta}_f \zeta_f}{H}. \quad (30)$$

The part of Lagrangian which gives the important contribution is the following;

$$\begin{aligned} g^2 \mathcal{L}_m &= \text{tr} \left\{ 2(D_\mu \phi)^\dagger D_\mu \phi + 2\sqrt{2}ig\lambda[\psi, \phi^\dagger] \right\} + (D_\mu Q)^\dagger D_\mu Q + D_\mu \tilde{Q}(D_\mu \tilde{Q})^\dagger \\ &\quad + \sqrt{2}i \left( \tilde{q}\phi q + Q^\dagger \lambda q - \tilde{q}\lambda \tilde{Q}^\dagger + \tilde{q}\psi Q + \tilde{Q}\psi q \right) \\ &= \partial_\mu \left\{ \text{tr}(2\phi^\dagger D_\mu \phi) + (D_\mu Q)^\dagger Q + (D_\mu \tilde{Q})^\dagger \tilde{Q} \right\} + \sqrt{2}i \left( \tilde{q}\phi q + Q^\dagger \lambda q + \tilde{Q}\psi q \right). \end{aligned} \quad (31)$$

The last equality follows from an integration by parts and the equation of supersymmetric instanton. To integrate the last term, we use the auxiliary solution  $\bar{q}$ ,

$$\bar{q}_{\dot{r}}^{\dot{\alpha}} = \frac{1}{4\pi} \epsilon_{\dot{r}\dot{s}} N^{\dagger \dot{s}r} \left\{ \mathcal{N}_r R^{-1} (\mathcal{M}^T)^s - \mathcal{M}_r R^{-1} (\mathcal{N}^T)^s + iF\delta_r^s \right\} M_{st} \epsilon^{t\dot{\alpha}} R^{-1} \tilde{\zeta}, \quad (32)$$

which satisfies the equation;

$$\not{D}\bar{q} + \sqrt{2}Q^\dagger \lambda + \sqrt{2}\tilde{q}\phi_f + \sqrt{2}\tilde{Q}\psi = -\Psi\eta, \quad (33)$$

where

$$\eta = \frac{1}{2}\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tilde{\zeta}. \quad (34)$$

Using the auxiliary solution  $\bar{q}$  and  $\phi_q^\dagger$ , the last term of Eq.(31) becomes

$$\partial_\mu \text{tr} \left\{ 2(D_\mu \phi_q^\dagger) \phi_0 \right\} - i\not{\partial}(\bar{q}q) - i\epsilon^{\alpha\beta} \epsilon_{\dot{r}\dot{s}} \Psi_{\dot{\beta}}^{\dot{s}} \eta q_{\dot{\alpha}}^{\dot{r}}. \quad (35)$$

From the normalization condition of  $\Psi$  and the asymptotic behaviors of  $\bar{q}$  and the supersymmetric instanton, the action of supersymmetric instanton becomes

$$\begin{aligned}
g^2 S &= 16\pi^2 + S_{higgs} + S_{yukawa}, \\
S_{higgs} &= 16\pi^2 \left( L|A_{00}|^2 - \frac{\omega^2}{H} \right), \\
S_{yukawa} &= -4\sqrt{2}\pi^2 \left\{ \nu_k A_{00} \mu_k + \frac{\omega}{H} (\mu_1 \nu_2 - \mu_2 \nu_1 + 2m_3 n_1 - 2m_1 n_3) \right\} \\
&\quad + \frac{1}{2H} (\mu_1 \nu_2 - \mu_2 \nu_1 + 2m_3 n_1 - 2m_1 n_3) \tilde{\zeta}_f \zeta_f + \sqrt{2} \frac{\omega}{H} \tilde{\zeta}_f \zeta_f.
\end{aligned} \tag{36}$$

Comparing to the pure Yang-Mills case, the last two term in  $S_{yukawa}$  are added. Note that a biquadratic term in Grassmannian variables appears in the action. This is a new feature in the N=2 supersymmetric QCD. The measure of the collective coordinate is given by[21, 14],

$$\begin{aligned}
C_J &\int d^4 x_0 d^4 a_3 d^4 \omega_1 d^4 \omega_2 d^2 \xi d^2 m_3 d^2 \mu_1 d^2 \mu_2 d^2 \xi' d^2 n_3 d^2 \nu_1 d^2 \nu_2 \\
&\times \prod_{f=1}^{N_f} d^2 \zeta_f d^2 \tilde{\zeta}_f \frac{|a_3|^2 - |a_1|^2}{H} \exp(-S_{higgs} - S_{yukawa}),
\end{aligned} \tag{37}$$

where the coupling constant  $g$  is absorbed by the redefinition of the collective coordinates and

$$C_J = 2^{6+2N_f} \pi^{-8} \Lambda_{N_f}^{8-2N_f}. \tag{38}$$

### 3 Instanton calculus

We calculate two-instanton contribution to  $\langle u \rangle = \langle \text{tr} \phi^2 \rangle$ . Taking into account the super transformation, it is easy to find that the adjoint scalar  $\phi$  contains the following part;

$$\phi = -\sqrt{2}i\xi\psi + \dots = \sqrt{2}i\xi\bar{\sigma}_{\mu\nu}\xi'F_{\mu\nu} + \dots, \tag{39}$$

where  $\dots$  includes the other fermionic zero modes and  $\phi_0$ . The normalization of supersymmetric modes,  $\xi$  and  $\xi'$  is determined by the Eq(9). Then,  $u$  is given by,

$$u = -2\text{tr} \left[ (\xi\bar{\sigma}_{\mu\nu}\xi'F_{\mu\nu})^2 \right] + \dots \tag{40}$$

$$= -\xi^2 \xi'^2 \text{tr} (F_{\mu\nu} F_{\mu\nu}) + \dots. \tag{41}$$

Therefore supersymmetric zero modes are saturated by inserting  $u$ , and we obtain the following result by performing the integration over the center of the instanton;

$$\int d^4 x_0 \int d^2 \xi d^2 \xi' u(x) = - \int d^4 x_0 \text{tr} [F_{\mu\nu}(x-x_0) F_{\mu\nu}(x-x_0)] = -32\pi^2. \tag{42}$$

The other fermionic modes are lifted by the Yukawa terms in the action, and integrating out those modes except  $\zeta_f, \tilde{\zeta}_f$ , we obtain

$$\begin{aligned} & \int d^2 m_3 d^2 \mu_1 d^2 \mu_2 d^2 n_3 d^2 \nu_1 d^2 \nu_2 \exp(-S_{yukawa}) \\ &= - \left( \frac{16\sqrt{2}\pi^6}{|a_3|^2 H |\Omega|} \right)^2 f(y) \exp\left(-\sqrt{2} \frac{\omega}{H} \tilde{\zeta}_f \zeta_f\right), \end{aligned} \quad (43)$$

where

$$f(y) = \omega^2 y^2 \left\{ \left( |\Omega|^2 |A_{00}|^2 + \frac{L\omega^2 y}{H} \right)^2 + \frac{L^2 - |\Omega|^2}{H^2} \omega^2 y^2 (|A_{00}|^2 |\Omega|^2 - \omega^2) \right\}, \quad (44)$$

$$y = 1 - \frac{\sqrt{2}}{16\pi^2 \omega} \tilde{\zeta}_f \zeta_f. \quad (45)$$

The remaining Grassmann integrations are performed in the following;

$$\int \prod_{f=1}^{N_f} d^2 \tilde{\zeta}_f d^2 \zeta_f f(y) \exp\left(-\sqrt{2} \frac{\omega}{H} \tilde{\zeta}_f \zeta_f\right) = \left(-\frac{1}{2} \frac{\omega^2}{H^2}\right)^{N_f} \sum_{k=0}^{2N_f} {}_{2N_f}C_k \left(\frac{H}{16\pi^2 \omega^2}\right)^k \left. \frac{\partial^k f}{\partial y^k} \right|_{y=1}. \quad (46)$$

We change the integration variables from  $a_3, \omega_1, \omega_2$  to  $H, L, \Omega$ , and then the measure of the integral becomes,

$$\int d^4 a_3 \frac{|a_3|^2 - |a_1|^2}{|a_3|^4} = \frac{\pi^2}{2} \int_{L+2|\Omega|}^{\infty} dH, \quad (47)$$

$$\int d^4 \omega_1 d^4 \omega_2 = \frac{\pi^3}{8} \int_0^{\infty} dL \int_{|\Omega| \leq L} d^3 \Omega. \quad (48)$$

With the change to a polar coordinate:  $\omega = |\Omega| |A_{00}| \cos \theta$  and the rescaling:  $\Omega' = \Omega/L$  and  $H' = H/L$ , the measure is given by,

$$\begin{aligned} & \frac{\pi^5}{16} \int_0^{\infty} dL \int_{|\Omega| \leq L} d^3 \Omega \int_{L+2|\Omega|}^{\infty} dH \\ &= \frac{\pi^6}{8} \int_0^{\infty} dL L^4 \int_{-1}^1 d(\cos \theta) \int_0^1 |\Omega'|^2 d|\Omega'| \int_{1+2|\Omega'|}^{\infty} dH', \end{aligned} \quad (49)$$

and  $f(y)$  becomes

$$f(y) = |A_{00}|^6 |\Omega'|^6 L^6 \cos^2 \theta G(y; |\Omega'|, H', \theta), \quad (50)$$

where

$$G(y; |\Omega'|, H', \theta) = y^2 \left\{ \left( 1 + \frac{y}{H'} \cos^2 \theta \right)^2 + \frac{1 - |\Omega'|^2}{4H'^2} y^2 \sin^2 2\theta \right\}. \quad (51)$$

Using Eq.(37), (42), (43), (46), (49) and (50) and performing the integration of  $L$ , we obtain the two-instanton correction to  $\langle u \rangle$ ,

$$\langle u \rangle_2 = \frac{1}{2} a^2 \left( \frac{\Lambda_{N_f}}{a} \right)^{8-2N_f} \cdot \left( -\frac{1}{2} \right)^{N_f} I(N_f), \quad (52)$$

where  $I(N_f)$  is defined by

$$I(N_f) = \int_{-1}^1 d(\cos \theta) \cos^2 \theta \int_0^1 d|\Omega'| |\Omega'|^6 \int_{1+2|\Omega'|}^\infty \frac{dH'}{H'^3} \left( \frac{|\Omega'| \cos \theta}{H'} \right)^{2N_f} \sum_{k=0}^K {}_{2N_f}C_k \quad (53)$$

$$\times (5-k)! \left( 1 - \frac{|\Omega'|^2 \cos^2 \theta}{H'} \right)^{k-6} \left( \frac{H'}{|\Omega'|^2 \cos^2 \theta} \right)^k \frac{\partial^k}{\partial y^k} G(y; |\Omega'|, H', \theta) \Big|_{y=1},$$

and  $K = \min[4, 2N_f]$ . The integral  $I(N_f)$  is complicated but elementary. Finally we obtain

$$\langle u \rangle_2 = \frac{1}{2} a^2 \times \begin{cases} \frac{5}{2} \left( \frac{\Lambda_0}{a} \right)^8 & \text{for } N_f = 0, \\ -\frac{3}{4} \left( \frac{\Lambda_1}{a} \right)^6 & \text{for } N_f = 1, \\ \frac{1}{8} \left( \frac{\Lambda_2}{a} \right)^4 & \text{for } N_f = 2, \\ -\frac{5}{2^4 3^3} \left( \frac{\Lambda_3}{a} \right)^2 & \text{for } N_f = 3. \end{cases} \quad (54)$$

## 4 Exact results versus instanton calculus

The low energy effective Lagrangians for  $N = 2$  supersymmetric gauge theories are determined by the holomorphic function  $\mathcal{F}$ , which is called the prepotential. According to [1], the prepotential  $\mathcal{F}$  are determined by the elliptic curves;

$$N_f = 0 \quad : \quad y^2 = x^2(x - u) + \frac{1}{4} \tilde{\Lambda}_0^4 x, \quad (55)$$

$$N_f = 1, 2, 3 \quad : \quad y^2 = x^2(x - u) - \frac{1}{64} \tilde{\Lambda}_{N_f}^{2(4-N_f)} (x - u)^{N_f-1}, \quad (56)$$

in the  $SU(2)$  gauge theories. In the semiclassical limit, prepotential  $\mathcal{F}$  is expanded by the one-loop correction and  $k$ -instanton contributions;

$$\mathcal{F}(a) = \frac{ia^2}{4\pi} \left\{ (4 - N_f) \ln \left( \frac{a^2}{\tilde{\Lambda}_{N_f}^2} \right) + \sum_{k=0}^{\infty} \mathcal{F}_k(N_f) \left( \frac{\tilde{\Lambda}_{N_f}}{a} \right)^{(4-N_f)k} \right\}. \quad (57)$$



In this convention, the coefficients  $\mathcal{F}_{2n+1}$  vanish for  $N_f \neq 0$ . The vacuum expectation value of  $u$  is given as the function of  $a$ [22] by,

$$\begin{aligned} u(a) &= \frac{8\pi i}{4 - N_f} \left( \mathcal{F}(a) - \frac{1}{2} a \partial_a \mathcal{F}(a) \right) \\ &= 2a^2 \left\{ 1 - \frac{1}{2} \sum_{k=1}^{\infty} k \mathcal{F}_k(N_f) \left( \frac{\tilde{\Lambda}_{N_f}}{a} \right)^{(4-N_f)k} \right\}. \end{aligned} \quad (58)$$

From the Picard-Fuchs equation, we can obtain  $\mathcal{F}_k$  recursively[23]. The Picard-Fuchs equation is given by,

$$p(u) \partial_a^2 u - a (\partial_a u)^3 = 0, \quad (59)$$

where

$$p(u) = \begin{cases} 4(u^2 - \tilde{\Lambda}_0^4) & \text{for } N_f = 0, \\ 4u^2 + \frac{27\tilde{\Lambda}_1^6}{64u} & \text{for } N_f = 1, \\ 4(u^2 - \frac{\tilde{\Lambda}_2^4}{64}) & \text{for } N_f = 2, \\ u(4u - \frac{\tilde{\Lambda}_3^2}{64}) & \text{for } N_f = 3. \end{cases} \quad (60)$$

Only when  $N_f=0$ , one-instanton contribution  $\mathcal{F}_1$  does not vanish, and this coefficient agrees with microscopic one-instanton calculus, if we identify the dynamical scale  $\tilde{\Lambda}_0 = \sqrt{2}\Lambda_0$ [12]. The two-instanton correction to  $\langle u \rangle$  is given by  $\mathcal{F}_2$ , and we obtain

$$\langle u \rangle_2 = 2a^2 \times \begin{cases} 5 \cdot 2^{-13} \left( \frac{\tilde{\Lambda}_0}{a} \right)^8 & \text{for } N_f = 0, \\ -3 \cdot 2^{-12} \left( \frac{\tilde{\Lambda}_1}{a} \right)^6 & \text{for } N_f = 1, \\ 2^{-11} \left( \frac{\tilde{\Lambda}_2}{a} \right)^4 & \text{for } N_f = 2, \\ 2^{-10} \left( \frac{\tilde{\Lambda}_3}{a} \right)^2 & \text{for } N_f = 3. \end{cases} \quad (61)$$

The relation between the dynamical scales  $\tilde{\Lambda}_{N_f}$  is given by,

$$m^2 \tilde{\Lambda}_{N_f}^{8-2N_f} = \tilde{\Lambda}_{N_f-1}^{8-2(N_f-1)}. \quad (62)$$

This decoupling relation also holds for  $\Lambda_{N_f}$ , which we have examined in the instanton calculus of the massive N=2 supersymmetric QCD. Using the relation  $\tilde{\Lambda}_0 = \sqrt{2}\Lambda_0$ , we obtain the relation between the dynamical scales:  $\tilde{\Lambda}_{N_f}^{8-2N_f} = 16\Lambda_{N_f}^{8-2N_f}$ . From this relation, we find that the microscopic instanton calculus agrees<sup>3</sup> with the exact results for  $N_f = 0, 1, 2$ . However we also find a discrepancy between them for  $N_f = 3$ .

In the similar way, we have evaluated the four-point function  $\langle \bar{\lambda}\bar{\lambda}\bar{\psi}\bar{\psi} \rangle$  by the instanton calculus[12, 14], and have found that the non-trivial relation Eq.(58) holds for  $N_f = 0, 1, 2$ . For  $N_f = 3$ , this four-point function does not depend on  $\mathcal{F}_2$ , therefore it is not useful to check the exact result.

More detail and complete explanations of this letter will appear in the near future[24].

## note added

1. After the completion of this work, we learned that the four point function  $\langle \bar{\lambda}\bar{\lambda}\bar{\psi}\bar{\psi} \rangle$  was calculated independently in [25].
2. The formulas (52) and (53) hold for  $N_f = 4$  by replacing  $\Lambda_{N_f}^{8-2N_f}$  with  $q = e^{-16\pi^2/g^2+2i\theta}$ . In this case,

$$\langle u \rangle_2 = \frac{1}{2}a^2 \times \frac{7}{2^5 3^5} q \quad .$$

This result does not agree with the exact result, which is based on the assumption that no quantum correction appears in this case. After completing this calculation, we received the paper [26], in which the quantum correction in the  $N_f = 4$  theory was discussed. We thank M. P. Mattis for informing us of the appearance of their preprint and also thank K. Ito and N. Sasakura for the discussion on this point.

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<sup>3</sup>Note on the convention for  $a$ : The definition of  $a$  differs between Eq.(54) and Eq.(61) by 2.

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